

Differential structure on the κ -Minkowski spacetime from twist

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We study four dimensional κ -Minkowski spacetime constructed by the twist deformation of $U(\text{igl}(4, R))$. We demonstrate that the differential structure of such twist-deformed κ -Minkowski spacetime is closed in four dimensions contrary to the construction of κ -Poincaré bicovariant calculus which needs an extra fifth dimension. Our construction holds in arbitrary dimensional spacetimes.

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1. Introduction

There has been much interest in recent years in a possible role of deformation of spacetime symmetry in describing Planck scale physics. In particular, initiated by the κ -deformed Poincaré algebra [1] the κ -Minkowski spacetime [2, 3] satisfying

$$[x^0, x^i] = \frac{i}{\kappa} x^i, \quad [x^i, x^j] = 0, \quad (1)$$

has attracted much attention in explaining cosmic observational data, since the deformation preserves the rotational symmetry in space. The differential structure of the κ -Minkowski spacetime has been constructed in [4] and based on this differential structure, the scalar field theory has been formulated [5, 6, 7]. Similar field theoretic approach is given in [8, 9] using the coproduct and star product as Lie-algebraic noncommutative spacetime. It was shown that the differential structure requires that the momentum space corresponding to the κ -Minkowski spacetime becomes a de-Sitter section in five-dimensional flat space. The κ -deformation was extended to the curved space with κ -Robertson-Walker metric and

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was applied to the cosmic microwave background radiation in [10]. The physical effects of the κ -deformation has been investigated on the Unruh effect [6], black-body radiation [11] and Casimir effect [12]. The Fock space and its symmetries [13, 14], κ -deformed statistics of particles [15, 16], and interpretation of the κ -Minkowski spacetime in terms of exotic oscillator [17] were also studied.

Recently, simpler realization of the κ -Minkowski spacetime by the use of twisting procedure have been sought by several authors [18, 19, 20, 21, 22]. It happens that only the case of the light-cone κ -deformation the deformed Poincaré algebra can be described by standard twist (see eg. [18]).

By embedding an abelian twist in $IGL(4, R)$ whose symmetry is larger than the Poincaré, the realization for the time-like κ -deformation was first constructed in Ref. [19] and then by [20]. Some physical properties of analogous twist realization of κ -Minkowski spacetime were discussed recently [21]. This approach can be seen as an alternative to the κ -like deformation of the quantum Weyl and conformal algebra [22], which is obtained by using the Jordanian twist [23]. One may even consider the chains of twists for classical Lie algebras [24].

In this letter, we will construct the κ -Minkowski spacetime and its differential structure using the twisted universal enveloping Hopf algebra of the inhomogeneous general linear group in (3+1)-dimensions. In section 2, the κ -Minkowski spacetime from twist is reviewed and in section 3, its differential structure is constructed.

2. Review on the κ -Minkowski spacetime from twist

Twisting the Hopf algebra of the universal enveloping algebra of $igl(4, R)$ is considered in [19, 20]. The group of inhomogeneous linear coordinate transformations is composed of the product of the general linear transformations and the spacetime translations. The inhomogeneous general linear algebra in (3+1)-dimensional flat spacetime $\mathfrak{g} = igl(4, R)$ is composed of 20 generators $\{P_a, M_b^a\}$ ($a, b = 0, 1, 2, 3$) where P_a represents the spacetime translation and M_b^a homogeneous one including the boost generator, rotation and dilation. The generators satisfy the commutation relations,

$$\begin{aligned} [P_a, P_b] &= 0, & [M_b^a, P_c] &= i\delta_c^a \cdot P_b, \\ [M_b^a, M_d^c] &= i(\delta_d^a \cdot M_b^c - \delta_b^c \cdot M_d^a). \end{aligned} \quad (2)$$

The universal enveloping Hopf algebra $\mathcal{U}(\mathfrak{g})$ can be constructed starting from the base elements $\{1, P_a, M_b^a\}$ and coproduct $\Delta Y = 1 \otimes Y + Y \otimes 1$ with $Y \in \{P_a, M_b^a\}$. The operators representing energy E and spatial dilatation D is defined by

$$E = P_0, \quad D = \sum_{i=1}^3 M_i^i. \quad (3)$$

Note that the two generators D and E commutes with each other, $[D, E] = 0$. In Ref. [19], an Abelian twist element \mathcal{F}_κ ,

$$\mathcal{F}_\kappa = \exp \left[\frac{i}{\kappa} \left(\alpha E \otimes D - (1 - \alpha) D \otimes E \right) \right], \quad (4)$$

is shown to generate the κ -Minkowski spacetime with the twisted Hopf algebra $\mathcal{U}_\kappa(\mathfrak{g})$. α is a constant chosen as $\alpha = 1/2$ in this letter which corresponds to the symmetric ordering

of the exponential kernel function in the conventional κ -Minkowski spacetime formulation. Other choice of α represents a different ordering.

Co-unit and antipode are not twisted $\epsilon_{\mathcal{F}} = \epsilon$ and $S_{\mathcal{F}} = S$, but coproduct is twisted as

$$\Delta_{\kappa}(Y) = \mathcal{F}_{\kappa} \cdot \Delta Y \cdot \mathcal{F}_{\kappa}^{-1} = \sum_i Y_{(1)i} \otimes Y_{(2)i} \equiv Y_{(1)} \otimes Y_{(2)}. \quad (5)$$

Explicitly, $(i, j = 1, 2, 3)$

$$\begin{aligned} \Delta_{\kappa}(Z) &= Z \otimes 1 + 1 \otimes Z, \quad Z \in \{E, D, M_0^0, M_j^i\}, \\ \Delta_{\kappa}(P_i) &= P_i \otimes e^{E/(2\kappa)} + e^{-E/(2\kappa)} \otimes P_i, \\ \Delta_{\kappa}(M_0^i) &= M_0^i \otimes e^{-E/(2\kappa)} + e^{E/(2\kappa)} \otimes M_0^i, \\ \Delta_{\kappa}(M_i^0) &= M_i^0 \otimes e^{E/(2\kappa)} + e^{-E/(2\kappa)} \otimes M_i^0 + \frac{1}{2\kappa} (P_i \otimes D e^{E/(2\kappa)} - e^{-E/(2\kappa)} D \otimes P_i). \end{aligned} \quad (6)$$

It is noted that the twisted Hopf algebra is different from that of the conventional κ -Poincaré algebra in two aspects. First, the algebraic part is nothing but those of the un-deformed inhomogeneous general linear group (2) rather than that of the deformed Poincaré. Second, the co-algebra structure is enlarged due to the bigger symmetry $igl(4)$ and its co-product is deformed as (6).

3. Differential structure

The inhomogeneous general linear group $IGL(4, R)$ acts on the coordinate space $\{x^a\}$ and the twisted-coproduct of the generator Y acts on the tensor product space of $\{x^a \otimes x^b\}$. Thus, one can define the $*$ -product of the coordinate vectors x^a in terms of the twist action on the coordinates. Explicitly,

$$x^a * x^b \equiv * [x^a \otimes x^b] = \cdot [\mathcal{F}_{\kappa}^{-1} \triangleright (x^a \otimes x^b)]. \quad (7)$$

This results in the noncommutative commutation relation of the coordinates

$$[x^0, x^j]_{\kappa} \equiv x^0 * x^j - x^j * x^0 = \frac{i}{\kappa} x^j, \quad [x_i, x_j]_{\kappa} = 0,$$

which reproduces the commutation relation (1).

To understand the differential structure, one has to incorporate the (co-)tangent space and investigate the action of $IGL(4, R)$ on the space. Suppose that one constructs a set of basis vectors of a coordinate system $CS = \{e_a | a = 0, 1, 2, 3\}$ of the four dimensional vector space V_4 which are not necessarily ortho-normal. One naturally demands that the homogeneous transformation Λ acts on the coordinates x^a , the dual-basis of the coordinate system e^a , and a function f as

$$\Lambda : \begin{cases} x^a \rightarrow x^{a'}; x^{a'} = x^b \Lambda_b^{a'}, \\ e^a \rightarrow e^{a'}; e^{a'} = e^b \Lambda_b^{a'}, \\ f \rightarrow f'; f'(x') = f(x) = f(x^{b'} (\Lambda^{-1})_{b'}^a). \end{cases} \quad (8)$$

and the translation T by the amount of coordinate vector y^a as

$$T(y^a) : \begin{cases} x^a \rightarrow x^{a'}; x^{a'} = x^a + y^a, \\ e^a \rightarrow e^{a'}; e^{a'} = e^a, \\ f \rightarrow f'; f'(x^{a'}) = f(x^a) = f(x^{a'} - y^a). \end{cases} \quad (9)$$

Then, the infinitesimal transformation is given in terms of $igl(4, R)$ generators:

$$\delta_\epsilon S = -i\epsilon^c Y_c \triangleright S. \quad (10)$$

The action of M_b^a is represented by

$$\begin{aligned} M_b^a \triangleright x^c &= -ix^a \delta_b^c, & M_b^a \triangleright e^c &= -ie^a \delta_b^c, \\ (M_b^a \triangleright f)(x^a) &= -ix^a \frac{\partial}{\partial x^b} f(x^a), \end{aligned} \quad (11)$$

and of P_a by

$$\begin{aligned} P_a \triangleright x^b &= -i\delta_a^b, & P_a \triangleright e^b &= 0, \\ (P_a \triangleright f)(x^b) &= -i \frac{\partial}{\partial x^a} f(x^b). \end{aligned} \quad (12)$$

Note that the translation and thus, the energy operator E does not change the dual basis vector e^a . On the other hand, the spatial dilatation operator D non-trivially acts as:

$$\begin{aligned} \exp(i\alpha D) \triangleright x^a &= x_{(\alpha)}^a \exp(i\alpha D) \triangleright, & \exp(i\alpha D) \triangleright e^a &= e_{(\alpha)}^a \exp(i\alpha D) \triangleright, \\ (\exp(i\alpha D) \triangleright f)(x^a) &= f(x_{(-\alpha)}^a), \end{aligned}$$

where $x_{(\alpha)}^a = (x^0, \exp(\alpha) x^i)$ and $e_{(\alpha)}^a = (e^0, \exp(\alpha) e^i)$. This non-trivial transformation law provides the $*$ -product between the space coordinates and/or the dual-basis vectors $\{x^a, e^a\}$. Between the two basis vectors, we have

$$e^a * e^b = \cdot [\mathcal{F}_\kappa^{-1}(e^a \otimes e^b)] = e^a e^b,$$

where the time translational invariance (12) $E \triangleright e^a = 0$ is used. Between e^a and x^b we have

$$\begin{aligned} e^a * x^b &= m[\mathcal{F}_\kappa^{-1}(e^a \otimes x^b)] = e^a x^b - \frac{i}{2\kappa} \delta_i^a \delta_0^b e^i, \\ x^b * e^a &= m[\mathcal{F}_\kappa^{-1}(x^b \otimes e^a)] = e^a x^b + \frac{i}{2\kappa} \delta_i^a \delta_0^b e^i, \end{aligned}$$

which results in the commutation relation

$$[e^a, e^b]_\kappa = 0, \quad [x^0, e^i]_\kappa = \frac{i}{\kappa} e^i, \quad [x^0, e^0]_\kappa = 0 = [x^i, e^a]_\kappa. \quad (13)$$

The twist deformation is also applied to the multiplication of two functions f and g which transform according to (8):

$$f(x) * g(x) \equiv m_\kappa[f \otimes g](x) := m[\mathcal{F}_\kappa^{-1}(f \otimes g)](x), \quad (14)$$

when the commutative multiplication is defined as $m[f \otimes g](x) := f(x)g(x)$. This twist deformation leads to the conventional κ -Minkowski star product,

$$f(x) * g(x) := \exp \left[\frac{i}{2\kappa} \left(\frac{\partial}{\partial x_0} y^k \frac{\partial}{\partial y_k} - x^k \frac{\partial}{\partial x_k} \frac{\partial}{\partial y_0} \right) \right] f(x)g(y) \Big|_{x=y}. \quad (15)$$

Explicitly, the star product of two exponential functions is given by

$$\begin{aligned}
e^{ipx} * e^{iqx} &= m \left(\exp \left[-\frac{i}{2\kappa} (E \otimes D - D \otimes E) \right] \triangleright (e^{ipx} \otimes e^{iqx}) \right) \\
&= m \left(\exp \left[-\frac{i}{2\kappa} (p_0 \otimes D - D \otimes q_0) \right] \triangleright (e^{ipx} \otimes e^{iqx}) \right) \\
&= e^{i(p_0+q_0)x^0 + (p_i e^{\frac{q_0}{2\kappa}} + q_i e^{-\frac{p_0}{2\kappa}})x^i}.
\end{aligned} \tag{16}$$

Note that this twist deformation reproduces the symmetric ordering result of the conventional κ -Minkowski spacetime [25]. It should be noted that the action of E and D applies on x -space only and not on p and q which are just numbers.

On the star-product, the action \triangleright_κ of $Y \in \{P_a, M_b^a\}$ is defined by

$$Y \triangleright_\kappa (A * B) = (Y_{(1)} \triangleright_\kappa A) * (Y_{(2)} \triangleright_\kappa B), \tag{17}$$

where $Y_{(1,2)}$ is defined in (5). The action \triangleright_κ on the right-hand side reduces to the undeformed one \triangleright when A or B contains no $*$ -product. In addition, the κ in \triangleright_κ only implies that the action is acting on the star product. In general, we may omit the κ .

We now elaborate on the differential calculus of the κ -Minkowski spacetime based on the twist deformation. The differential structure constructed in Ref. [4] are five dimensional, which was based on Jacobi identity and the usual Leibnitz rule. Here we will *not require the usual* Leibnitz rule. Instead, we identify the partial derivative with the generator element iP_a . In this procedure, Leibnitz rule is naturally modified by the twist operation and four dimensional differential calculus is obtained.

The derivative $\partial_a = iP_a$ on the $*$ -products is defined according to (17) with $\Delta_\kappa(\partial_a) = i\Delta_\kappa(P_a)$ and is governed by the rule,

$$\begin{aligned}
\partial_0 \triangleright_\kappa (\phi(x) * \psi(x)) &= (\partial_0 \triangleright_\kappa \phi(x)) * \psi(x) + \phi(x) * (\partial_0 \triangleright_\kappa \psi(x)), \\
\partial_i \triangleright_\kappa (\phi(x) * \psi(x)) &= (\partial_i \triangleright_\kappa \phi(x)) * \left(e^{\frac{E}{2\kappa}} \psi(x) \right) + \left(e^{-\frac{E}{2\kappa}} \phi(x) \right) * (\partial_i \triangleright_\kappa \psi(x)),
\end{aligned} \tag{18}$$

which gives the modified Leibnitz rule and is different from the one in [4].

Our definition of the derivative (18) has following nice properties. First, the derivative rule is consistent with the $*$ -product. Suppose one acts the derivative on the exponential functions e^{ipx} in two ways. One using the translation rule (12): $P_a \triangleright e^{ipx} = p_a e^{ipx}$. The other is to use the symmetric ordering $[e^{ipx}]_s \equiv e^{ip_0 x^0/2} * e^{i\vec{p} \cdot \vec{x}} * e^{ip_0 x^0/2}$ and use the modified Leibnitz rule (18): $P_a \triangleright_\kappa [e^{ipx}]_s = P_a \triangleright_\kappa (e^{ip_0 x^0/2} * e^{i\vec{p} \cdot \vec{x}} * e^{ip_0 x^0/2}) = p_a [e^{ipx}]_s$. For the action of the generator M_b^a , we also have $M_b^a \triangleright e^{ipx} = p_b x^a e^{ipx}$ and $M_b^a \triangleright_\kappa [e^{ipx}]_s = p_b \triangleright_\kappa (e^{ip_0 x^0/2} * (x^a e^{i\vec{p} \cdot \vec{x}}) * e^{ip_0 x^0/2})$.

Second, one can define the differential operator d_κ acting on a function f or a n -form $\omega = dx^{a_1} \wedge dx^{a_2} \cdots \wedge dx^{a_n} f_{a_1 a_2 \cdots a_n}$

$$\begin{aligned}
d_\kappa f &= dx^a * (iP_a \triangleright_\kappa f), \\
d_\kappa \omega &= dx^c \wedge dx^{a_1} \wedge dx^{a_2} \cdots \wedge dx^{a_n} * (iP_c \triangleright_\kappa f_{a_1 a_2 \cdots a_n}), \\
d_\kappa f \wedge d_\kappa g &= -d_\kappa g \wedge d_\kappa f.
\end{aligned} \tag{19}$$

In this calculation, we implicitly assume that the wedge include the $*$ -product. (See the comment below (23).) The definition of d_κ allows differential calculus on the $*$ -product.

Explicitly, one has

$$\begin{aligned}
d_\kappa(x^0 * x^i) &= id x^a * [P_a \triangleright_\kappa (x^0 * x^i)] = dx^0 * x^i + dx^j * (e^{-E/(2\kappa)} x^0 * i(P_j \triangleright x^i)) \quad (20) \\
&= (dx^0) * x^i + (dx^i) * x^0 + \frac{i}{2\kappa} dx^i, \\
d_\kappa(x^i * x^0) &= id x^a * [P_a \triangleright_\kappa (x^i * x^0)] = dx^0 * x^i + dx^j * (i(P_j \triangleright x^i) * e^{E/(2\kappa)} x^0) \\
&= (dx^0) * x^i + (dx^i) * x^0 - \frac{i}{2\kappa} dx^i.
\end{aligned}$$

On the other hand, in the differential geometry, the coordinate representation of e^a is given by dx^a . Therefore, from Eq. (13), we have commutation relations with differential elements

$$[dx^a, dx^b]_\kappa = 0, \quad [x^0, dx^i]_\kappa = \frac{i}{\kappa} dx^i, \quad [x^0, dx^0]_\kappa = 0 = [x^i, dx^a]_\kappa. \quad (21)$$

This relations are consistent with d_κ operation:

$$d_\kappa[x^0, x^i]_\kappa = \frac{i}{\kappa} dx^i = [dx^0, x^i]_\kappa + [x^0, dx^i]_\kappa$$

from (20) even though the Leibnitz rule (18) in general shows $d_\kappa[f, g]_\kappa \neq [df, g]_\kappa + [f, dg]_\kappa$.

Third, one can also check that the general linear transformation acting on both side of Eq. (21) is consistent with the coproduct (6). Especially,

$$M_l^0 \triangleright_\kappa [x^i, dx^j]_\kappa = M_l^0 \triangleright_\kappa (x^i * dx^j - dx^j * x^i) = 0, \quad (22)$$

which is ensured by the equations $[x^0, dx^j]_\kappa = \frac{i}{\kappa} dx^j$ and $[x^i, dx^0]_\kappa = 0$, and the second equation in (11). This result is also consistent with $M_b^a \triangleright_\kappa (dx^c) = d(M_b^a \triangleright_\kappa x^c)$, which comes from the postulate that the action of the general linear algebra extends to the differential algebra in a natural covariant way [4]. The relations (21) has been also proposed in [13, 26], however the action of κ -Poincaré boost generators show that the relation (22) is not satisfied [4].

Finally, the last equation of (19) and the commutativity of $P_a \triangleright_\kappa$ operations ($P_a \triangleright_\kappa P_b \triangleright_\kappa = P_b \triangleright_\kappa P_a \triangleright_\kappa$) ensure the identity

$$d_\kappa^2 \omega = 0 \quad (23)$$

and makes the $*$ -product with the wedge in (19) be irrelevant. One may also confirm that the Jacobi identity such as

$$[x^a, [x^b, dx^c]_\kappa]_\kappa + [x^b, [dx^c, x^a]_\kappa]_\kappa + [dx^c, [x^a, x^b]_\kappa]_\kappa = 0$$

is satisfied with the present differential structure (21).

In conclusion, our proposal provides the κ -covariance of the differential calculus (21) without introducing an extra dimensional differential. We have enlarged the carrier algebra of the twist function (from Poincaré to $IGL(4, R)$) but we preserve the classical algebra. This is in contrast with the proposal in [22] which is based on much smaller extension of Poincaré algebra (from Poincaré to Weyl) but the classical algebra is also modified. It will be interesting to find the differential structure explicitly in this approach.

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